DIAGONAL COINVARIANTS AND DOUBLE AFFINE HECKE ALGEBRAS

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Contents

It was conjectured by Haiman [H] that the space of diagonal coinvariants for a root system R of rank n has a "natural" quotient of dimension $(1+h)^n$ for the Coxeter number h. This space is the quotient $\mathbb{C}[x,y]/(\mathbb{C}[x,y]\mathbb{C}[x,y]^W_o)$ for the algebra of polynomials $\mathbb{C}[x,y]$ with the diagonal action of the Weyl group on $x \in \mathbb{C}^n \ni y$ and the ideal $\mathbb{C}[x,y]^W_o \subset \mathbb{C}[x,y]^W$ of the W-invariant polynomials without the constant term. In [G], such a quotient was constructed. It appeared to be the graded object of the perfect module (in the terminology of [C9]) of the rational double affine Hecke algebra for the simplest nontrivial $k=-1-1/h$.

Generally, the perfect modules are defined as irreducible self-dual spherical representations of DAHA with a projective action of the $PSL_2(\mathbb{Z})$. In the q-case, the semisimplicity is added. At roots of unity, they generalize the Verlinde algebras. Gordon gives an explicit description of the above module as a quotient of the space of double polynomials considered as a representation of the rational DAHA induced from the sign-character of the nonaffine Weyl group W.

Date: May 15, 2003.

† Partially supported by NSF grant DMS-0200276.

In [C9], perfect modules appear naturally as quotients of the algebra of single polynomials. Using the double polynomials has some advantages. For instance, the self-duality becomes obvious. We note that the perfect modules are always quotients of the space of double polynomials, but the corresponding kernels are expected to be reasonably simple only for $k \in -1/h - \mathbb{Z}_+$.

We extend Gordon's description to the q-case, establishing its *direct* connection with a fundamental fact that the Weyl algebra of rank n (a noncommutative n-torus) has a unique irreducible representation provided that the center element q is a primitive N-th root of unity and the generators are cyclic of order N. Its dimension is N^n, which matches the Haiman number as $N = 1 + h$. We *deduce* our theorem from this fact and, as a corollary, obtain a new, entirely algebraic, proof of Gordon's theorem using the Lusztig-type isomorphism acting from the general DAHA to its rational degeneration.

Gordon's demonstration was based on the results due to Opdam–Rouquier (see [GGOR]) on the monodromy of the KZ-connection from [C1],[C3] in relation to the representation theory of the rational DAHA. The technique of Lusztig's isomorphisms is actually of the same origin. These isomorphisms are closely connected with the monodromy of the *affine* KZ-connection.

The structure of the paper is as follows. We start with the general definitions and the construction of the Lusztig-type isomorphisms (for reduced root systems). Then we switch to the case of generic q and $k = -1 - 1/h$. Following [C8],[C9], we make q a root of unity. In [C8], this method was used to deform the Verlinde algebras. Finally, we obtain our theorem and reprove (the main part of) the theorem from [G].

Lusztig's isomorphisms are important by themselves. Under minor restrictions, they establish an equivalence of the categories of finite dimensional representations of DAHA when q is not a root of unity and those of its rational degeneration. They can be applied to infinite dimensional representations as well, however, generally speaking, the theory gets analytic.

The last section of the paper contains the definitions of the following two new objects, the *universal* double affine Hecke algebra and the corresponding universal Dunkl operators acting in the *noncommutative* polynomials in terms of two sets of variables X and Y. Upon the reduction to the commutative polynomials, these operators are directly connected with the main theorem and have other applications. We note that the universal DAHA satisfies a "noncommutative" variant of the PBW–theorem.

The definition of the universal DAHA is $X \leftrightarrow Y$–symmetric as well as its homomorphism to the DAHA. The $X \leftrightarrow Y$–duality of the DAHA was deduced in [C2] from the topological interpretation of the double affine ("elliptic") braid group. A direct proof of the DAHA-duality is not difficult too (see [M]). The universal DAHA, to be more exact the corresponding braid group, can be used to simplify the proof. It has something in common with the method from [IS].

Concerning the elliptic braid group and its topological interpretation, there is a connection with the construction due to v.d.Lek, although the orbifold fundumental group was used in [C2] insead of removing the ramification divisor in his construction. This connection was mentioned in [C2] and is discussed in more detail in [Io]. In the case of GL_n, the braid group from [C2] is essentially due to Birman and Scott.

The author is thankful to P. Etingof for important remarks and discussions. The paper was completed at Institut de Mathématiques de Luminy. I am grateful for the invitation. I also thank the referee for a thorough report.

1. Double affine Hecke algebras

Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form (z, z') on $\mathbb{R}^n \ni z, z'$, W the Weyl group generated by the reflections s_α, R_+ the set of positive roots, corresponding to (fixed) simple roots roots $\alpha_1, ..., \alpha_n$, Γ the Dynkin diagram with $\{\alpha_i, 1 \le i \le n\}$ as the vertices, $R^\vee = \{\alpha^\vee = 2\alpha/(\alpha, \alpha)\}$ the dual root system,

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for the simple coroots $\alpha_i^\vee$.

The form will be normalized by the condition $(\alpha, \alpha) = 2$ for the *short* roots. This normalization coincides with that from the tables in [B] for A, C, D, E, G. Hence $\nu_\alpha \overset{\mathbf{def}}{=\!=} (\alpha, \alpha)/2$ can be $1, 2$ or 3. Sometimes we write ν_{lng} for long roots ($\nu_{\mathrm{sht}} = 1$). Let $\vartheta \in R^\vee$ be the maximal positive *coroot* (it is maximal short in R), $\rho = (1/2)\sum_{\alpha\in R_+}\alpha \ = \sum_i \omega_i$.

Affine roots. The vectors $\tilde\alpha = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R, j \in \mathbb{Z}$ form the *affine root system* $\tilde R \supset R$ ($z \in \mathbb{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \stackrel{\mathbf{def}}{=\!=} [-\vartheta, 1]$ to the simple roots. The set $\tilde R$ of positive roots is $R_+ \cup \{[\alpha, \nu_\alpha j],\ \alpha \in R,\ j > 0\}$. Let $\tilde\alpha^\vee = \tilde\alpha/\nu_\alpha$, so $\alpha_0^\vee = \alpha_0$.

The Dynkin diagram Γ of R is completed by α_0 (by $-\vartheta$ to be more exact). The notation is $\tilde\Gamma$. It is the completed Dynkin diagram for $R^\vee$ from [B] with the arrows reversed.

The set of the indices of the images of α_0 by all the automorphisms of $\tilde\Gamma$ will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O' = r \in O, r \neq 0$. The elements ω_r for $r \in O'$ are the so-called minuscule weights: $(\omega_r, \alpha^\vee) \leq 1$ for $\alpha \in R_+$.

Given $\tilde\alpha = [\alpha, \nu_\alpha j] \in \tilde R,\ b \in P$, let

$$
s_{\tilde\alpha}(\tilde z) \ = \ \tilde z - (z, \alpha^\vee)\tilde\alpha, \quad b(\tilde z) \ = \ [z, \zeta - (z, b)] \tag{1.1}
$$

for $\tilde z = [z, \zeta] \in \mathbb{R}^{n+1}$.

The *affine Weyl group* $\widetilde W$ is generated by all $s_{\tilde\alpha}$. One can take the simple reflections $s_i = s_{\alpha_i}$ $(0 \leq i \leq n)$ as its generators and introduce the corresponding notion of the length. This group is the semidirect product $W \ltimes Q$ of its subgroups W and the lattice Q, where $\alpha \in Q$ is identified with $s_\alpha s_{[\alpha,\nu_\alpha]} = s_{[-\alpha,\nu_\alpha]} s_\alpha$ for $\alpha \in R$.

The *extended Weyl group* $\widehat W$ generated by W and P is isomorphic to $W \ltimes P$:

$$
(wb)([z, \zeta]) \ = \ [w(z), \zeta - (z, b)] \ \text{ for } \ w \in W, b \in P. \tag{1.2}
$$

Given $b \in P_+$, let w_0^b be the longest element in the subgroup $W_0^b \subset W$ of the elements preserving b. This subgroup is generated by simple reflections. We set

$$
u_b = w_0 w_0^b \in W, \ \pi_b = b(u_b)^{-1} \ \in \ \widehat W, \ u_i = u_{\omega_i}, \pi_i = \pi_{\omega_i}, \tag{1.3}
$$

where w_0 is the longest element in W, $1 \leq i \leq n$.

The elements $\pi_r \stackrel{\mathbf{def}}{=\!=} \pi_{\omega_r}, r \in O'$ and $\pi_0 = \mathrm{id}$ leave $\tilde\Gamma$ invariant and form a group denoted by Π, which is isomorphic to P/Q by the natural projection $\{\omega_r \mapsto \pi_r\}$. As to $\{u_r\}$, they preserve the set $\{-\vartheta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (u_r)^{-1}(-\vartheta)$ distinguish the indices $r \in O'$.

Moreover,

$$(1.4)\quad \widehat{W} = \Pi \ltimes \widetilde{W}, \quad \text{where} \quad \pi_r s_i \pi_r^{-1} = s_j \quad \text{if} \quad \pi_r(\alpha_i) = \alpha_j,\ 0 \le j \le n.$$

Setting $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$, $\pi_r \in \Pi$, $\widetilde{w} \in \widetilde{W}$, the *length* $l(\widehat{w})$ is by definition the length of the reduced decomposition $\widetilde{w} = s_{i_l} ... s_{i_2} s_{i_1}$ in terms of the simple reflections $s_i, 0 \le i \le n$.

DAHA. By m, we denote the least natural number such that $(P,P) = (1/m)\mathbb{Z}$. Thus $m = 2$ for D_{2k}, $m = 1$ for B_{2k}, C_k, otherwise $m = |\Pi|$.

The double affine Hecke algebra depends on the parameters q, t_ν, $\nu \in \{\nu_\alpha\}$. The definition ring is $\mathbb{Q}_{q,t} \stackrel{\mathbf{def}}{=\!=} \mathbb{Q}[q^{\pm 1/m}, t^{\pm 1/2}]$ formed by the polynomials in terms of $q^{\pm 1/m}$ and $\{t_\nu^{\pm 1/2}\}$. We set

$$\begin{aligned} & t_{\tilde{\alpha}} = t_\alpha = t_{\nu_\alpha},\ t_i = t_{\alpha_i},\ q_{\tilde{\alpha}} = q^{\nu_\alpha},\ q_i = q^{\nu_{\alpha_i}}, \\ (1.5)\qquad & \text{where} \ \ \tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R},\ 0 \le i \le n. \end{aligned}$$

It will be convenient to use the parameters $\{k_\nu\}$ together with $\{t_\nu\}$, setting

$$t_\alpha = t_\nu = q_\alpha^{k_\nu} \quad \text{for} \quad \nu = \nu_\alpha, \quad \text{and} \quad \rho_k = (1/2) \sum_{\alpha > 0} k_\alpha \alpha.$$

For pairwise commutative $X_1, \ldots, X_n$,

$$\begin{aligned} (1.6)\qquad & X_{\tilde{b}} \ = \ \prod_{i=1}^n X_i^{l_i} q^j \ \text{ if } \ \tilde{b} = [b, j],\ \widehat{w}(X_{\tilde{b}}) \ = \ X_{\widehat{w}(\tilde{b})}. \\ & \text{where } b = \sum_{i=1}^n l_i \omega_i \in P,\ j \in \frac{1}{m}\mathbb{Z},\ \widehat{w} \in \widehat{W}. \end{aligned}$$

Later $Y_{\tilde{b}} = Y_b q^{-j}$ will be needed. Note the opposite sign of j. We set $(\tilde{b}, c) = (b, c)$.

We will also use that π_r^{-1} is π_{r^*} and u_r^{-1} is u_{r^*} for $r^* \in O$, $u_r = \pi_r^{-1}\omega_r$. The reflection * is induced by an involution of the nonaffine Dynkin diagram Γ.

Definition 1.1. *The double affine Hecke algebra* $\mathcal{H}$ *is generated over* $\mathbb{Q}_{q,t}$ *by the elements* $\{T_i,\ 0 \le i \le n\}$, *pairwise commutative* $\{X_b,\ b \in P\}$ *satisfying (1.6), and the group* Π, *where the following relations are imposed:*

(o) $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) \ = \ 0,\ 0 \ \le \ i \ \le \ n;$

(i) $T_i T_j T_i ... \ = \ T_j T_i T_j ...,\ m_{ij}$ *factors on each side;*

(ii) $\pi_r T_i \pi_r^{-1} \ = \ T_j$ *if* $\pi_r(\alpha_i) = \alpha_j$;
(iii) $T_i X_b T_i \ = \ X_b X_{\alpha_i}^{-1}$ *if* $(b, \alpha_i^\vee) = 1, \ 0 \le i \le n$;
(iv) $T_i X_b \ = \ X_b T_i$ *if* $(b, \alpha_i^\vee) = 0$ *for* $0 \le i \le n$;
(v) $\pi_r X_b \pi_r^{-1} \ = \ X_{\pi_r(b)} \ = \ X_{u_r^{-1}(b)} q^{(\omega_{r^*}, b)}, \ r \in O'$.

□

Given $\widetilde{w} \in \widetilde{W}, r \in O,$ the product

$$T_{\pi_r \widetilde{w}} \stackrel{\mathbf{def}}{=\!=} \pi_r \prod_{k=1}^{l} T_{i_k}, \quad \text{where} \quad \widetilde{w} = \prod_{k=1}^{l} s_{i_k}, l = l(\widetilde{w}), \tag{1.7}$$

does not depend on the choice of the reduced decomposition (because $\{T\}$ satisfy the same "braid" relations as $\{s\}$ do). Moreover,

$$T_{\hat{v}} T_{\widehat{w}} \ = \ T_{\hat{v}\widehat{w}} \ \text{ whenever } l(\hat{v}\widehat{w}) = l(\hat{v}) + l(\widehat{w}) \ \text{ for } \ \hat{v}, \widehat{w} \in \widehat{W}. \tag{1.8}$$

In particular, we arrive at the pairwise commutative elements

$$Y_b \ = \ \prod_{i=1}^{n} Y_i^{l_i} \ \text{ if } \ b = \sum_{i=1}^{n} l_i \omega_i \in P, \ \text{ where } \ Y_i \stackrel{\mathbf{def}}{=\!=} T_{\omega_i}, \tag{1.9}$$

satisfying the relations

$$\begin{aligned} T_i^{-1} Y_b T_i^{-1} \ &= \ Y_b Y_{\alpha_i}^{-1} \ \text{ if } \ (b, \alpha_i^\vee) = 1, \\ T_i Y_b \ &= \ Y_b T_i \ \text{ if } \ (b, \alpha_i^\vee) = 0, \ 1 \le i \le n. \end{aligned} \tag{1.10}$$

For arbitrary nonzero q, t, any element $H \in \mathcal{H\!\!H}$ has a unique decomposition in the form

$$H = \sum_{w \in W} g_w \, f_w \, T_w, \ g_w \in \mathbb{Q}_{q,t}[X], \ f_w \in \mathbb{Q}_{q,t}[Y], \tag{1.11}$$

and five more analogous decompositions corresponding to the other orderings of $\{T, X, Y\}$. It makes the polynomial representation (to be defined next) the $\mathcal{H\!\!H}$-module induced from the one dimensional representation $T_i \mapsto t_i^{1/2}, \ Y_i \mapsto Y_i^{1/2}$ of the affine Hecke subalgebra $\mathcal{H}_Y = \langle T, Y \rangle$.

These and below statements are from [C4].

One may also use the *intermediate subalgebras* of $\mathcal{H\!\!H}$ with P replaced by any lattice $B \ni b$ between Q and P for X_b and Y_b (see [C9]). Respectively, Π is changed to the preimage of B/Q in Π. Generally, there can be two different lattices B_x and B_y for X and Y. The $m \in \mathbb{N}$ from the definition of $\mathbb{Q}_{q,t}$ has to be the least such that $m(B_x, B_y) \subset \mathbb{Z}$.

Note that $\mathcal{H}$, its degenerations, and the corresponding polynomial representations are actually defined over $\mathbb{Z}$ extended by the parameters of DAHA. We will use its $\mathbb{Z}$-structure a couple of times in the paper (the modular reduction), but prefer to stick to $\mathbb{Q}$. The Lusztig isomorphisms require $\mathbb{Q}$.

Demazure-Lusztig operators. They are defined as follows:

$$T_i \ = \ t_i^{1/2} s_i \ + \ (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1}(s_i - 1), \ 0 \le i \le n, \tag{1.12}$$

and obviously preserve $\mathbb{Q}[q, t^{\pm 1/2}][X]$. We note that only the formula for T_0 involves q:

$$\begin{aligned} T_0 \ &= t_0^{1/2} s_0 \ + \ (t_0^{1/2} - t_0^{-1/2})(qX_\vartheta^{-1} - 1)^{-1}(s_0 - 1), \\ \text{where } \ s_0(X_b) \ &= \ X_b X_\vartheta^{-(b,\vartheta)} q^{(b,\vartheta)}, \ \alpha_0 = [-\vartheta, 1]. \end{aligned} \tag{1.13}$$

The map sending T_j to the formula from (1.12), $X_b \mapsto X_b$ (see (1.6)), $\pi_r \mapsto \pi_r$ induces a $\mathbb{Q}_{q,t}$-linear homomorphism from $\mathcal{H}$ to the algebra of linear endomorphisms of $\mathbb{Q}_{q,t}[X]$. This $\mathcal{H}$-module, which will be called the *polynomial representation*, is faithful and remains faithful when q, t take any nonzero complex values assuming that q is not a root of unity.

The images of the Y_b are called the *difference Dunkl operators*. To be more exact, they must be called trigonometric-difference Dunkl operators, because there are also rational-difference Dunkl operators.

Automorphisms. Assuming that $B_x = B_y$, the following maps can be uniquely extended to automorphisms of $\mathcal{H}$ (see [C5],[C9]):

$$\varepsilon : \ X_i \mapsto Y_i, \ Y_i \mapsto X_i, \ T_i \mapsto T_i^{-1} \, (i \ge 1), \, t_\nu \mapsto t_\nu^{-1}, \, q \mapsto q^{-1}, \tag{1.14}$$

$$\begin{aligned} &\tau_+ : X_b \mapsto X_b, \, Y_r \mapsto X_r Y_r q^{-\frac{(\omega_r, \omega_r)}{2}}, \ T_i \mapsto T_i \, (i \ge 1), \ \ t_\nu \mapsto t_\nu, \ q \mapsto q, \\ &\tau_+ : Y_\vartheta \mapsto q^{-1} \, X_\vartheta T_0^{-1} T_{s_\vartheta}, \, T_0 \mapsto q^{-1} \, X_\vartheta T_0^{-1}, \ \text{ and} \end{aligned} \tag{1.15}$$

$$\tau_- \ \stackrel{\mathbf{def}}{=\!=} \varepsilon \tau_+ \varepsilon, \ \text{ and } \ \sigma \stackrel{\mathbf{def}}{=\!=} \tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1} = \varepsilon \sigma^{-1} \varepsilon, \tag{1.16}$$

where $r \in O'$. In the definition of $\tau_\pm$ and σ, we need to add $q^{\pm 1/(2m)}$ to $\mathbb{Q}_{q,t}$. Here the quadratic relation (o) from Definition 1.1 may be omitted. Only the group relations matter. The elements $\tau_\pm$ generate the projective $PSL(2, \mathbb{Z})$, which is isomorphic to the braid group B_3 due to Steinberg.

Intertwining operators. The Y*-intertwiners* (see [C6]) are introduced as follows:

$$
\begin{aligned}
\Phi_i \;&=\; T_i + (t_i^{1/2} - t_i^{-1/2})(Y_{\alpha_i}^{-1} - 1)^{-1} \ \text{ for } \ 1 \le i \le n,\\
\Phi_0 \;&=\; X_\vartheta T_{s_\vartheta} - (t_0^{1/2} - t_0^{-1/2})(Y_0 - 1)^{-1},\ Y_0 = Y_{\alpha_0} \stackrel{\mathbf{def}}{=\!=} q^{-1} Y_\vartheta^{-1},
\end{aligned}
$$

$$
G_i = \Phi_i(\phi_i)^{-1},\ \phi_i = t_i^{1/2} + (t_i^{1/2} - t_i^{-1/2})(Y_{\alpha_i}^{-1} - 1)^{-1}. \tag{1.17}
$$

Actually these formulas are the ε-images of the formulas for the X-intertwiners, which are a straightforward generalization of those in the affine Hecke theory.

They belong to $\mathcal{H\!\!H}$ extended by the rational functions in terms of $\{Y\}$. The G are called the *normalized intertwiners.* The elements

$$
G_i,\ P_r \stackrel{\mathbf{def}}{=\!=}\ X_r T_{u_r^{-1}},\ 0 \le i \le n,\ r \in O',
$$

satisfy the same relations as $\{s_i, \pi_r\}$ do, so the map

$$
\widehat{w} \mapsto G_{\widehat{w}} \;=\; P_r G_{i_l} \cdots G_{i_1}, \ \text{ where } \ \widehat{w} = \pi_r s_{i_l} \cdots s_{i_1} \in \widehat{W}, \tag{1.18}
$$

is a well defined homomorphism from $\widehat{W}$.

The intertwining property is

$$
G_{\widehat{w}} Y_b G_{\widehat{w}}^{-1} = Y_{\widehat{w}(b)} \ \text{ where } \ Y_{[b,j]} \stackrel{\mathbf{def}}{=\!=} Y_b q^{-j}.
$$

The P_1 in the case of GL is due to Knop and Sahi.

As to Φ_i, they satisfy the homogeneous Coxeter relations and those with Π_r. So we may set $\Phi_{\widehat{w}} = P_r \Phi_{i_l} \cdots \Phi_{i_1}$ for the reduced decompositions. They intertwine Y as well.

The formulas for Φ_i when $1 \le i \le n$ are well known in the theory of affine Hecke algebras. The affine intertwiners are the raising operators for the Macdonald nonsymmetric polynomials, serve the Harish-Chandra – Opdam spherical transform, and are the key tool in the theory of semisimple representations of DAHA.

2. Degenerate DAHA

Recall that $m(P,P) \in \mathbb{Z}$ or $m(B,B) \in \mathbb{Z}$ if B is used,

$$
k_i = k_{\alpha_i},\ k_0 = k_{\mathrm{sht}},\ \nu_\alpha = (\alpha,\alpha)/2 \in \{1,2,3\}.
$$

We set $\mathbb{Q}_k \stackrel{\mathbf{def}}{=\!=} \mathbb{Q}[k_\alpha]$. If the integral coefficients are needed, we take $\mathbb{Z}_k \stackrel{\mathbf{def}}{=\!=} \mathbb{Z}[k_\alpha, 1/m]$ as the definition ring.

The *degenerate (graded) double affine Hecke algebra* $\mathcal{H}\!\!\!\mathcal{H}'$ is the span of the group algebra $\mathbb{Q}_k[\widehat{W}]$ and the pairwise commutative

$$y_{\tilde{b}} \stackrel{\mathbf{def}}{=\!=} \sum_{i=1}^{n}(b,\alpha_i^\vee)y_i + u \quad \text{for} \quad \tilde{b} = [b,u] \in P \times \mathbb{Z},$$

satisfying the following relations:

$$\begin{aligned} s_j y_b - y_{s_j(b)} s_j &= -k_j(b,\alpha_j),\ (b,\alpha_0) \stackrel{\mathbf{def}}{=\!=} -(b,\vartheta), \\ \pi_r y_{\tilde{b}} &= y_{\pi_r(\tilde{b})}\pi_r \quad \text{for} \quad 0 \le j \le n,\ r \in O. \end{aligned} \tag{2.1}$$

Comment. Without s_0 and π_r, we arrive at the defining relations of the graded affine Hecke algebra from [L]. The algebra $\mathcal{H}\!\!\!\mathcal{H}'$ has two natural polynomial representations via the differential-trigonometric and difference-rational Dunkl operators. There is also the third one, a representation in terms of infinite differential-trigonometric Dunkl operators, which leads to differential-elliptic W-invariant operators generalizing those due to Olshanetsky- Perelomov. See, e.g., [C6]. We will need here only the (most known) differential-trigonometric polynomial representations. $\square$

Let us establish the connection with the general DAHA. We set

$$q = \exp(\mathfrak{v}),\ t_j = q_i^{k_i} = q^{\nu_{\alpha_i} k_i},\ Y_b = \exp(-\mathfrak{v} y_b),\ \mathfrak{v} \in \mathbb{C}.$$

Using ε from (1.14), the algebra $\mathcal{H}\!\!\!\mathcal{H}$ is generated by $Y_b,\ T_i$ for $1 \le i \le n$, and

$$\varepsilon(T_0) = X_\vartheta T_{s_\vartheta},\ \varepsilon(\pi_r) = X_r T_{u_r^{-1}},\ r \in O'.$$

It is straightforward to see that the relations (2.1) for y_b, $s_i (i > 0)$, s_0, π_r are the leading coefficients of the $\mathfrak{v}$-expansions of the general relations for this system of generators. Thus $\mathcal{H}\!\!\!\mathcal{H}'$ is $\mathcal{H}\!\!\!\mathcal{H}$ in the limit $\mathfrak{v} \to 0$.

When calculating the limits of the Y_b in the polynomial representation, the "trigonometric" derivatives of $\mathbb{Q}[X]$ appear:

$$\partial_a(X_b) \ = \ (a,b)X_b,\ a,b \in P,\ \ w(\partial_b) = \partial_{w(b)},\ w \in W.$$

The Y_b result in the *trigonometric Dunkl operators*

$$\mathcal{D}_b \stackrel{\mathbf{def}}{=\!=} \partial_b + \sum_{\alpha\in R_+} \frac{k_\alpha(b,\alpha)}{(1 - X_\alpha^{-1})}\bigl(1 - s_\alpha\bigr) - (\rho_k, b). \tag{2.2}$$

They act on the Laurent polynomials $f \in \mathbb{Q}_k[X]$, are pairwise commutative, and $y_{[b,u]} = \mathcal{D}_b + u$ satisfy (2.1) for the following action of the group $\widehat{W}$:

$$w^x(f) = w(f) \ \text{ for } \ w \in W, \ b^x(f) = X_b f \ \text{ for } \ b \in P.$$

For instance, $s_0^x(f) = X_\vartheta s_\vartheta(f), \ \pi_r^x(f) = X_r u_r^{-1}(f)$.

Degenerating $\{\Phi\}$, one gets the intertwiners of $\mathcal{H}'$:

$$\Phi_i' = s_i + \frac{\nu_i k_i}{y_{\alpha_i}}, \ 0 \le i \le n, \ \big(\, \Phi_0' = X_\vartheta s_\vartheta + \frac{k_0}{1 - y_\vartheta} \text{ in } \mathbb{Q}_k[X] \,\big),$$

$$P_r' = \pi_r, \ \big(\, P_r' = X_r u_r^{-1} \text{ in } \mathbb{Q}_k[X] \,\big), \ r \in O'. \tag{2.3}$$

The operator P_1' in the case of GL (it is of infinite order) plays the key role in [KS].

Recall that the general normalized intertwiners are

$$G_i = \Phi_i \phi_i^{-1}, \ \phi_i = t^{1/2} + (t_i^{1/2} - t^{1/2})(Y_{\alpha_i}^{-1} - 1)^{-1}.$$

Their limits are

$$G_i' = \Phi_i'(\phi_i')^{-1}, \ \phi_i' = 1 + \frac{\nu_i k_i}{y_{\alpha_i}}.$$

They satisfy the unitarity condition $(G_i')^2 = 1$, and the products $G_{\widehat{w}}'$ can be defined for any decompositions of $\widehat{w}$. One has:

$$G_{\widehat{w}}' \, y_b \, (G_{\widehat{w}}')^{-1} \ = \ y_{\widehat{w}(b)}.$$

Equating

$$G_i = G_i' \ \text{ for } \ 0 \le i \le n, \ P_r = P_r' \ \text{ for } \ r \in O,$$

we come to the formulas for $T_i \, (0 \le i \le n), \ X_r \, (r \in O')$ in terms of $s_i, y_b, Y_b = \exp(-\mathfrak{v} y_b)$.

These formulas determine the *Lusztig homomorphism* $\text{æ}'$ from $\mathcal{H}$ to the completion $\mathbb{Z}_{k,q,t}\mathcal{H}'[[\mathfrak{v} y_b]]$ for $\mathbb{Z}_{k,q,t} \stackrel{\mathbf{def}}{=\!=} \mathbb{Z}_k \mathbb{Z}_{q,t}$. See, e.g., [C6].

For instance, $X_r \in \mathcal{H}$ becomes $\pi_r T^{-1}_{u_r^{-1}}$ in $\mathcal{H}'$, where the T-factor has to be further expressed in terms of s, y. In the degenerate polynomial representation, $\text{æ}'(X_r)$ acts as $X_r(\text{æ}'(T_{u_r^{-1}}) u_r)^{-1}$, not as the straightforward multiplication by X_r. They coincide only in the limit $\mathfrak{v} \to 0$, when T_w become w.

Upon the $\mathfrak{v}$-completion, we get an isomorphism

$$\text{æ}' : \mathbb{Q}_k[[\mathfrak{v}]] \otimes \mathcal{H} \to \mathbb{Q}_k[[\mathfrak{v}]] \otimes \mathcal{H}'.$$

We will use the notation $(d,[\alpha,j])=j$. For instance, $(b+d,\alpha_0)=1-(b,\vartheta)$.

Treating $\mathfrak{v}$ as a nonzero number, an arbitrary $\mathcal{H}'$-module V' which is a union of finite dimensional Y-modules has a natural structure of an $\mathcal{H}$-module provided that we have

$$
\begin{aligned}
(2.4)\qquad & q^{(\alpha_i,\xi+d)} = t_i \Rightarrow (\alpha_i,\xi+d)=\nu_i k_i,\\
& q^{(\alpha_i,\xi+d)} = \ 1 \Rightarrow (\alpha_i,\xi+d)=0, \quad \text{where}\\
& 0\le i\le n,\ y_b(v')=(b,\xi)v' \ \text{ for } \ \xi\in\mathbb{C}^n,\ 0\ne v'\in V'.
\end{aligned}
$$

For the modules of this type, the map $\text{æ}'$ is over the ring $\mathbb{Q}_{k,q,t}$ extended by $(\alpha,\xi+d)$, $q^{(\alpha,\xi+d)}$ for $\alpha\in R$ and y-eigenvalues ξ. Moreover, we need to localize by $(1-q^{(\alpha,\xi+d)})\ne 0$ and by $(\alpha,\xi+d)\ne 0$. Upon such extension and localization, $\text{æ}'$ is defined over $\mathbb{Z}_{k,q,t}$ if the module is y-semisimple. If there are nontrivial Jordan blocks, then the formulas will contain factorials in the denominators.

For instance, let $\mathcal{I}'[\xi]$ be the $\mathcal{H}'$-module induced from the one-dimensional y-module $y_b(v)=(b,\xi)v$. Assuming that q is not a root of unity, the mapping $\text{æ}'$ supplies it with a structure of $\mathcal{H}$-module if

$$q^{(\alpha,\xi)+\nu_\alpha j}=t_\alpha \text{ implies } (\alpha,\xi)+\nu_\alpha j=\nu_\alpha k_\alpha$$

for every $\alpha\in R, j\in\mathbb{Z}$, and the corresponding implications hold for t replaced by 1. This means that

$$(2.5)\qquad (\alpha,\xi)-\nu_\alpha k_\alpha,\ (\alpha,\xi)\ \notin\ \nu_\alpha\mathbb{Z}+\frac{2\pi\imath}{\mathfrak{v}}(\mathbb{Z}\setminus\{0\}) \text{ for all } \alpha\in R.$$

Generalizing, $\text{æ}'$ is well defined for any $\mathcal{H}'$-module generated by its y-eigenvectors with the y-eigenvalues ξ satisfying this condition, assuming that $\mathfrak{v}\notin\pi\imath\mathbb{Q}$.

Comment. Actually there are at least four different variants of $\text{æ}'$ because the normalization factors ϕ,ϕ' may be associated with different one dimensional characters of the affine Hecke algebra $\langle T,Y\rangle$ and its degeneration. There is also a possibility to multiply the normalized intertwiners by the characters of $\widehat{W}$ before equating. Note that if we divide the intertwiners Φ and/or Φ' by ϕ,ϕ' on the left in the definition of G,G', it corresponds to switching from $T_i\mapsto t_i$ to the character $T_i\mapsto -t_i^{-1/2}$ together with the multiplication by the sign-character of $\widetilde{W}$. In the paper, we will use only $\text{æ}'$ introduced above. $\square$

Rational degeneration. The limit to the Dunkl operators is as follows. We set $X_b = e^{\mathfrak{w}x_b}$, $d_b(x_c) = (b, c)$, so the above derivatives ∂_b become $\partial_b = (1/\mathfrak{w})d_b$. In the limit $\mathfrak{w} \to 0$, $\mathfrak{w}\mathcal{D}_b$ tends to

$$D_b \stackrel{\mathbf{def}}{=\!=} d_b + \sum_{\alpha \in R_+} \frac{k_\alpha(b, \alpha)}{x_\alpha}\big(1 - s_\alpha\big). \tag{2.6}$$

These operators are pairwise commutative and satisfy the cross-relations

$$D_b x_c - x_c D_b = (b, c) + \sum_{\alpha>0} k_\alpha(b, \alpha)(c, \alpha^\vee)s_\alpha, \quad \text{for} \quad b, c \in P. \tag{2.7}$$

These relations, the commutativity of D, the commutativity of x, and the W-equivariance

$$w\, x_b\, w^{-1} \;=\; x_{w(b)},\;\; w\, D_b\, w^{-1} \;=\; D_b \quad \text{for} \;\; b \in P,\, w \in W,$$

are the defining relations of the *rational DAHA* $\mathcal{H}''$.

The references are [CM] (the case of A_1) and [EG], however the key part of the definition is the commutativity of D_b due to Dunkl [D]. The Dunkl operators and the operators of multiplication by the x_b form the *polynomial representation* of $\mathcal{H}''$, which is faithful. It readily justifies the PBW-theorem for $\mathcal{H}''$.

Note that in contrast to the q, t-setting, the definition of the rational DAHA can be extended to finite groups generated by complex reflections (Dunkl, Opdam, Malle). There is also a generalization due to Etingof- Ginzburg from [EG] (the symplectic reflection algebras).

Comment. Following [CO], there is a one-step limiting procedure from $\mathcal{H}$ to $\mathcal{H}''$. We set

$$Y_b = \exp(-\sqrt{\mathfrak{u}}D_b),\;\; X_b = e^{\sqrt{\mathfrak{u}}x_b},$$

assuming that $q = e^{\mathfrak{u}}$ and tend $\mathfrak{u} \to 0$. We come directly to the relations of the rational DAHA and the formulas for D_b. The advantage of this direct construction is that the automorphisms $\tau_\pm$ obviously survive in the limit. Indeed, τ_+ in $\mathcal{H}$ can be interpreted as the formal conjugation by the q-Gaussian $q^{x^2/2}$, where $x^2 = \sum_i x_{\omega_i} x_{\alpha_i^\vee}$. In the limit, it becomes the conjugation by $e^{x^2/2}$, preserving $w \in W$, x_b, and taking D_b to $D_b - x_b$. Respectively, τ_- preserves w and D_b, and sends $x_b \mapsto x_b - D_b$. These automorphisms do not exist in the $\mathcal{H}'$. $\square$

The *abstract Lusztig map* from $\mathcal{H}'$ to $\mathcal{H}''$ is as follows. Let $w \mapsto w$. We expand X_α in terms of x_α in the formulas for the trigonometric

Dunkl operators $\mathcal{D}_b$:

$$\mathcal{D}_b = \frac{1}{\mathfrak{w}} D_b - (\rho_k, b) + \sum_{\alpha \in R_+} k_\alpha (b, \alpha) \sum_{m}^{\infty} \frac{B_m}{m!} (-\mathfrak{w} x_\alpha)^m \, (1 - s_\alpha) \tag{2.8}$$

for the Bernoulli numbers B_m. Then we can use them as abstract expressions for y_b in terms of the generators of $\mathcal{H}''$.

One obtains an isomorphism $\text{æ}'' : \mathbb{Q}[[\mathfrak{w}]] \otimes \mathcal{H}' \to \mathbb{Q}[[\mathfrak{w}]] \otimes \mathcal{H}''$, which maps $\mathcal{H}'$ to the extension of $\mathcal{H}''$ by the formal series in terms of $\mathfrak{w} x_b$. An arbitrary representation V'' of $\mathcal{H}''$ which is a union of finite dimensional $\mathbb{Q}_k[x]$-modules becomes an $\mathcal{H}'-$ module provided that

$$\mathfrak{w}\zeta_\alpha \notin 2\pi\imath(\mathbb{Z} \setminus \{0\}) \ \text{ for } \ x_b(v) = (\zeta, b)v, 0 \neq v \in V''. \tag{2.9}$$

Similar to (2.5), this constraint simply restricts choosing $\mathfrak{w} \neq 0$. The formulas for y_b become locally finite in any representations of $\mathcal{H}''$ where x_b act locally nilpotent, for instance, in finite dimensional $\mathbf{H}''$-modules. In this case, there are no restrictions for $\mathfrak{w}$.

Finally, the composition

$$\text{æ} \stackrel{\mathbf{def}}{=\!=} \text{æ}'' \circ \text{æ}' : \mathcal{H}[[\mathfrak{v}, \mathfrak{w}]] \to \mathcal{H}''[[\mathfrak{v}, \mathfrak{w}]]$$

is an isomorphism. Without the completion, it makes an arbitrary finite dimensional $\mathcal{H}''$-module V'' a module over $\mathcal{H}$ as $q = e^{\mathfrak{v}}, t_\alpha = q^{k_\alpha}$ for sufficiently general (complex) nonzero numbers $\mathfrak{v}, \mathfrak{w}$. Note that isomorphism was discussed in [BEG] (Proposition 7.1).

The finite dimensional representations are the most natural here because, on one side, $\text{æ}''$ lifts the modules which are unions of finite dimensional x-modules to those for X, on the other side, $\text{æ}'$ maps the $\mathcal{H}'$-modules which are unions of finite dimensional y-modules to those for Y. So one must impose these conditions for both x and y. Using æ for infinite dimensional representations is an interesting problem. It makes the theory analytic. For instance, the triple composition $\text{æ}'' \circ \mathcal{G} \circ \text{æ}'$ for the inverse Opdam transform $\mathcal{G}$ (see [O] and formula (6.1) from [C7]) embeds $\mathcal{H}$ in $\mathcal{H}''$ and identifies the $\mathcal{H}''$-module $\mathbb{C}_c^\infty(\mathbb{R}^n)$ with the $\mathcal{H}$-module of PW-functions under the condition $\Re k > -1/h$. See [O, C7] for more detail.

Gordon's theorem. Let $k_{\text{sht}} = -(1 + 1/h) = k_{\text{lng}}$, h be the Coxeter number $1 + (\rho, \vartheta)$. The polynomial representation $\mathbb{Q}[x]$ of $\mathcal{H}''$ is generated by 1 and is $\{D, W\}$-spherical (1 is the only W-invariant polynomial killed by all D_b). Therefore it has a unique nonzero irreducible quotient-module. It is of dimension $(1+h)^n$, which was checked in [BEG], [G], and also follows from [C9] via **æ**.

Actually, a natural setting for Gordon's theorem is with the parameters $k_\nu = -(e_\nu + 1/h)$ for arbitrary integers $e_{\mathrm{sht}}, e_{\mathrm{lng}} \geq 0$. The W-invariants and W-antiinvariants of the corresponding perfect modules are connected by the shift operators. Cf. Conjecture 7.3 from [BEG]. Such k will not be considered in the paper.

The application of this representation to the coinvariants of the ring of commutative polynomials $\mathbb{Q}[x,y]$ with the diagonal action of W is as follows.

The polynomial representation $\mathbb{Q}[x]$ is naturally a quotient of the *linear space* $\mathbb{Q}[x,y]$ considered as an induced $\mathcal{H}''$-module from the one dimensional W-module $w \mapsto 1$. So is V''. The subalgebra $(\mathcal{H}'')^W$ of the W-invariant elements from $\mathcal{H}''$ preserves $\mathbb{Q}\delta \subset V''$ for $\delta \stackrel{\mathbf{def}}{=\!=} \prod_{\alpha>0} x_\alpha$.

Let $I_o \subset (\mathbf{H}'')^W$ be the ideal of the elements vanishing at the image of δ in V''. Gordon proves that V'' *coincides with the quotient* $\tilde{V}''$ *of* $\mathcal{H}''(\delta)$ *by the* $\mathcal{H}''$*-submodule* $\mathcal{H}'' I_o(\delta)$. It is sufficient to check that $\tilde{V}''$ is irreducible.

The graded space $\mathrm{gr}(V'')$ of V'' with respect to the total x,y-degree of the polynomials is isomorphic as a linear space to the quotient of $\mathbb{Q}[x,y]\delta$ by the graded image of $\mathcal{H}'' I_o(\delta)$. The latter contains $\mathbb{Q}[x,y]_o^W \delta$ for the ideal $\mathbb{Q}[x,y]_o^W \subset \mathbb{Q}[x,y]^W$ of the W-invariant polynomials without the constant term. Therefore V'' becomes a certain quotient of $\mathbb{Q}[x,y]/(\mathbb{Q}[x,y]\mathbb{Q}[x,y]_o^W)$. See [G],[H] about the connection with the Haiman theorem in the A-case and related questions for other root systems.

The irreducibility of the $\tilde{V}''$ above is the key fact. The proof from [G] requires considering the KZ-type local systems. We demonstrate that the irreducibility can be readily proved in the q,t- case using the passage to the roots of unity and therefore gives an entirely algebraic and simple proof of Gordon's theorem via the **æ**-isomorphism.

We note that the q,t-generalization V of V'' is in many ways simpler than V''. For instance, $\dim(V)$ can be readily calculated. However the filtration of V'' with respect to the degree of polynomials is a special feature of the rational limit as well as the character formula from [BEG, G], although the corresponding resolution has a q,t- counterpart. We will consider next the special situation when $1+h=p$ is prime and the definition field is $\mathbf{F}_p$ (see below). In this case, the required filtration can be defined in the q,t-setting as well as for the rational degeneration.

Modular reduction. Concerning the same problems over $\mathbb{Z}$, we may define the polynomial representation and V'' over p-adic numbers $\mathbb{Z}_p$ and take its fiber over $\mathbf{F}_p = \mathbb{Z}/(p)$. There is an interesting example when $1+h$ is a prime number p. Then $k=0$, and $\mathbf{F}_p \otimes \mathcal{H}''$ becomes the algebra of differential operators in x with the coefficients in $\mathbf{F}_p$. Its unique irreducible representation over F_p with the nilpotent action of x, y is the space $\mathbf{F}_p[x]/(x^p)$ of dimension p^n.

This proves the theorem from [G] for such h. Indeed, the module $\mathbf{F}_p \otimes \tilde{V}''$ is a semisimple W-module which contains a unique one dimensional submodule with the character $w \mapsto \operatorname{sgn}(w)$. Therefore it has to coincide with $\mathbf{F}_p[x]/(x^p)$. Since $\mathbf{F}_p \otimes \tilde{V}''$ is irreducible, so is $\tilde{V}''$.

An immediate application of this construction is that the space of diagonal coinvariants modulo $p = 1+h$ has a "natural" quotient of dimension p^n isomorphic to $\mathbf{F}_p \otimes V''$.

Actually our general prove has something in common with this argument. However it goes via the roots of unity instead of the modular reduction and holds for arbitrary h.

3. General case

The t-counterpart of the element δ is

$$\Delta = \prod_{\alpha \in R_+} (t_\alpha^{1/2} X_\alpha^{1/2} - t_\alpha^{-1/2} X_\alpha^{-1/2}).$$

It plays the key role in the definition of the t-shift operator (see [C4]). One has:

$$T_i(\Delta) = -t_i^{-1/2} \Delta, \ 1 \le i \le n.$$

We extend $\varpi_-(T_i) = -t_i^{-1/2}$ to a one dimensional representation of the nonaffine Hecke algebra $\mathbf{H}$ generated by $T_i, 1 \le i \le n$.

Coming to a q,t-generalization of Gordon's theorem, let q be generic, $k_{\text{lng}} = -(1+1/h) = k_{\text{sht}}$ for the Coxeter number h. We denote the field of rationals of $\mathbb{Q}_{q,t}$ by $\tilde{\mathbb{Q}}_{q,t}$.

Theorem 3.1. *i) The polynomial representation $\mathbb{Q}_{q,t}[X]$ of $\mathcal{H}$ has a unique nonzero quotient V which is torsion free and irreducible over $\tilde{\mathbb{Q}}_{q,t}$. It is of dimension $(1+h)^n$. The action of X and Y is semisimple* *with simple spectra. The module $V\otimes\tilde{\mathbb{Q}}_{q,t}$ considered as an $\mathbf{H}$-module contains a unique submodule isomorphic to ϖ_-.*

ii) The module V coincides with the quotient $\tilde V$ of $\mathbb{Q}_{q,t}[X]$ by the $\mathcal{H}$-submodule $\mathcal{H}\mathcal{I}_o(\tilde{\mathbb{Q}}_{q,t}\Delta)$ intersected with $\mathbb{Q}_{q,t}[X]$, where $\mathcal{I}_o$ is the kernel of the algebra homomorphism $\mathcal{H}_{inv}\ni H\mapsto H(\Delta)\in V$ for the subalgebra $\mathcal{H}_{inv}$ of the elements of $\mathcal{H}$ commuting with $T_1,\ldots,T_n$.

Proof. Concerning the existence of V and its isomorphism with the space $\mathrm{Funct}[P/(1{+}h)P]$, see Theorem 8.5 and formula (8.32) from [C9]. The description there is for general $k=-r/h$ as $(r,h)=1$. Actually we need here only the self-duality of V, that is, the action of the involution ε of $\mathcal{H}$ from (1.14) in V. It readily follows from the realization of V as the quotient of $Q_{q,t}[X]$ by the radical of the invariant bilinear form from [C9], Lemma 8.3. In fact, the self-duality will be needed only upon the specialization $\bullet$ below.

We mention that the automorphisms $\tau_\pm$ can be defined in V as well, but we do not use it in the paper.

The construction of V in [C9] holds over $\mathbb{Q}_{q,t}$. Actually it suffices to have the definition of V and to prove the theorem over $\tilde{\mathbb{Q}}_{q,t}$. Then one can use the standard facts about the modules over PID. Note that the module $\mathbb{Q}_{q,t}[X]/\mathcal{H}\mathcal{I}_o(\Delta)$, generally speaking, has torsion.

The uniqueness of ϖ_- in $V\otimes\tilde{\mathbb{Q}}_{q,t}$ results from the following fact:

$W(\rho)$ is a unique simple W-orbit in $P/(1+h)P$,

which will be checked below.

Comment. There is another proof based on the shift operator, which identifies the ϖ_--component of $V=V^k$ with the ϖ_+-component of V^{k+1} defined for $k+1=-1/h$, where $\varpi_+: T_i\mapsto t_i^{1/2}$. The V^{k+1} is one dimensional and coincides with its ϖ_+-component. The shift operator here is the division by Δ. This description is convenient to calculate the ideal $\mathcal{I}_o$. Its intersections with $\mathbb{Q}_{q,t}[X]_{\mathrm{inv}}$ and $\mathbb{Q}_{q,t}[Y]_{\mathrm{inv}}$ are not difficult to describe.

The self-dualty of V combined with the uniqueness of ϖ_- in V give formally that $\tilde V$ is self-dual too. Indeed, the character ϖ_- is ε-invariant. So are $\mathbb{Q}_{q,t}\Delta\subset V$, $\mathcal{I}_o$, and the kernel of the map $\mathbb{Q}_{q,t}[X]\to\tilde V$. The explicit discriminant formula for Δ is not helpful in checking that $\varepsilon(\Delta)$ is proportional to Δ in V.

The self-duality of $\tilde V$ can be also deduced from the ε-invariance of $\mathcal{I}_o$, which can be seen directly. $\square$

For $N \stackrel{\mathbf{def}}{=} 1+h$, we take $q=\exp(2\pi\imath/N)$ making $k=0, t^{\pm 1/2}=1$. Using that ν_α and the index of P/Q are relatively prime to N, we will pick $q^{1/m}$ in the roots of unity of the same order N. The $^\bullet$ will be used to denote this specialization, The algebra $\mathcal{HH}^\bullet$ is nothing else but the semidirect product of the Weyl algebra generated by pairwise commutative X_a, Y_b for $a,b\in P$ and $\mathbf{H}^\bullet=\mathbb{Q}_qW$. The relations are

$$w\,X_a\,w^{-1}=X_{w(a)},\; w\,Y_b\,w^{-1}=Y_{w(b)},\;\; X_aY_bX_a^{-1}Y_b^{-1}=q^{-(a,b)}.$$

We define $V^\bullet$ as a unique nonzero irreducible quotient of $\mathbb{Q}_q[X]$. It is self-dual. It results from [C9]. However it is straightforward to check it directly in the $\bullet$-case as well as the semisimplicity of X and Y.

Since all eigenvalues of Y_b in $V^\bullet$ (and in the whole $\mathbb{Q}_q[X]$) are N-th roots of unity, the same holds for X_b in $V^\bullet$ thanks to the self-duality. Thus $X_b^N=1=Y_b^N$ in $V^\bullet$ for all $b\in P$.

Theorem 8.5 from [C9] guarantees that V remains irreducible under such reduction, so $V^\bullet$ is the specialization of V. It can be also seen from the dimension formula in the following lemma.

Lemma 3.2. *i) The algebra $\mathcal{HH}^\bullet_N \stackrel{\mathbf{def}}{=} \mathcal{HH}^\bullet/(X^N=1=Y^N)$ has a unique irreducible nonzero representation $V^\bullet$ up to isomorphisms. Its dimension is N^n.*

ii) It is isomorphic to $Q_q[P/NP]$ as a W-module. The representation $\varpi^\bullet_- : w\mapsto sgn(w)$ has multiplicity one in $V^\bullet$.

iii) The quotient $\tilde{V}^\bullet$ of $\mathbb{Q}_q[X]$ by $\mathcal{HH}^\bullet\mathcal{I}^\bullet_o(\Delta^\bullet)$ is an $\mathcal{HH}^\bullet_N$-module and coincides with $V^\bullet$.

Proof. In the first place, $\mathcal{HH}^\bullet_N$ is a group algebra of a finite group, therefore semisimple. Let us use the well known fact that the Weyl algebra generated by X_a, Y_b modulo the (central) relations $X^N=1=Y^N$ has a unique irreducible representation up to isomorphisms and W acts in this representation. This representation equals $\mathbb{Q}_q[X]/(X^N=1)$ and is nothing else but $V^\bullet$.

The multiplicity one statement from ii) follows from the uniqueness of a simple W-orbit in P/NP. Let us check it.

We can assume that the orbit is $W(b)$ for $b\in P$ such that $0<(b,\alpha^\vee)<1+h$ for all $\alpha\in R_+$. Therefore b can be $\rho=\sum_i^n\omega_i$ or $\rho+\omega_r$ for $r\in O'$, i.e., for a minuscule weight ω_r. Indeed, the coefficient of $\alpha_i^\vee$ in the decomposition of ϑ in terms of simple coroots is one only for

$r \in O'$. For $b = \rho + \omega_r$, let $w = u_r^{-1}$ for u_r from $\omega_r = \pi_r u_r$. Then

$$(w(\rho) - b, \alpha_r) = -(\rho, \vartheta) - (\omega_r, \alpha_r^\vee) - 1 = -(1 + h) = -N \quad \text{and}$$
$$(w(\rho) - b, \alpha_i) = (\rho, \alpha_j) - (\rho, \alpha_i) = 0 \quad \text{for} \quad i \neq r, \ w^{-1}(\alpha_i) = \alpha_j.$$

Thus b and ρ generate the same W-orbit modulo NP.

The module $V^\bullet$ is self-dual. So is $\tilde{V}^\bullet$ because the $\mathbf{H}$-module $\mathbb{Q}_q\Delta^\bullet$ is of multiplicity one in $\tilde{V}^\bullet$ and therefore invariant with respect to ε. It gives that $\tilde{V}^\bullet$ is a finite dimensional $\mathcal{H\!\!H}_N^\bullet$-module. It has $V^\bullet$ as a quotient, and contains a unique W-submodule isomorphic to $\varpi_-^\bullet$.

Supposing that the kernel K of the map $\tilde{V}^\bullet \to V^\bullet$ is nonzero, it must contain a nonzero $\mathcal{H\!\!H}_N^\bullet$-submodule. Hence K contains at least one copy of $V^\bullet$ (the uniqueness), and the multiplicity of $\varpi_-^\bullet$ in $\tilde{V}^\bullet$ cannot be one. $\square$

We would like to mention that claims (ii)–(iii) are somewhat unusual in the general theory of Weyl algebras. Given the rank, they hold only for special choice of roots of unity. For instance, N must be 3 in the case of A_1.

Coming back to the general case, the coincidence statement of the theorem is actually over $\tilde{\mathbb{Q}}_{q,t}$, so it suffices to check it at one special point. The lemma gives it for the $\bullet$-point. To be more exact, claim (iii) of the lemma gives the irreducibility of $\tilde{V}$ and therefore the coincidence $\tilde{V} = V$ at the common point and the theorem. $\square$

The applications to the diagonal coinvariants goes via the universal Dunkl operators, which will be introduced in the next section. The problem is to calculate the action of Y_b in the linear space of Laurent polynomials $\mathbb{Q}_{q,t}[X, Y]$ identified with the $\mathcal{H\!\!H}$-module induced from a one dimensional character of $\mathbf{H}$. We assume here that the X-monomials are placed before Y-monomials. Then the action of X_b in the space $\mathbb{Q}_{q,t}[X, Y]$ is the "commutative" multiplication by X_b. The action of Y_b is by the left multiplication of the monomials in the form $X^\cdot Y^\cdot$. Hence it requires reordering and leads to nontrivial formulas.

The $\mathcal{H\!\!H}$-module $\mathbb{Q}_{q,t}[X, Y]$ is obviously self-dual. However since it is necessary to order X and Y after applying ε, the formulas for its action in $\mathbb{Q}_{q,t}[X, Y]$ are involved.

Now, the module V is the quotient of the module $\mathbb{Q}_{q,t}^-[X, Y]$ induced for the character ϖ_- of $\mathbf{H}$ by its submodule $\mathcal{H\!\!H}\mathcal{I}_o(1)$. It is a one-parametric deformation of the polynomial ring $\mathbb{Q}_{q,t}[X, Y]$ divided by the ideal $\mathbb{Q}_{q,t}[X,Y]^W_o = \{g(X,Y)(f(X,Y)-f(1,1))\}$ for W-invariant Laurent polynomials f and arbitrary g. Therefore it can be identified with a quotient of the space of diagonal coinvariants

$$\mathbb{Q}[X,Y]/(\mathbb{Q}[X,Y]\mathbb{Q}[X,Y]^W_o) \simeq \mathbb{Q}[x,y]/(\mathbb{Q}[x,y]\mathbb{Q}[x,y]^W_o),$$

with the ring of definition extended to $\mathbb{Q}_{q,t}$.

Concerning Gordon's theorem, the degenerations V', V'' of V can be introduced as quotients of the polynomial representation by the radicals of the degenerations of the bilinear form from [C9], Lemma 8.3. They are irreducible modules for $\mathcal{HH}', \mathcal{HH}''$ thanks to Lusztig's isomorphisms, and satisfy the same multiplicity one statement. The modules $\tilde{V}', \tilde{V}''$ are defined in terms of the ideal of the W-invariant elements of the double Hecke algebra vanishing at the discriminant subspace of V', V'' (corresponding to the sign-character of W). The $\tilde{V}', \tilde{V}''$ are irreducible due to Lusztig's isomorphisms, and therefore $V' = \tilde{V}'$ and $V'' = \tilde{V}''$. The latter is the (main part of the) Gordon's theorem.

One can also consider the specialization $\mathcal{HH}^\bullet$ assuming that N is a prime number p and make the field of constants $\mathbf{F}_p$. Then the space $\mathbf{F}_p[X,Y]/(\mathbf{F}_p[X,Y]\mathbf{F}_p[X,Y]^W_o)$ has a "natural" quotient-space isomorphic to $V^\bullet$ over $\mathbf{F}_p$. In this case, the formulas for the Lusztig homomorphism **æ** contain no denominators divisible by p and it is well defined over $\mathbf{F}_p$. Applying **æ**, we establish the coincidence of this quotient with the one obtained at the end of Section 2 using the differential operators over $\mathbf{F}_p$.

4. Universal DAHA

First, we will give a $X \leftrightarrow Y$–symmetric presentation of $\mathcal{HH}$. It goes via the *universal double affine braid group* $\widehat{\mathfrak{B}}$. This group is defined to be generated by the pairwise commutative X_b, the pairwise commutative Y_b, the elements $\widehat{T}_i$, where $b \in P$, $0 \le i \le n$, and the group $\widehat{\Pi} = \{\widehat{\pi}_r, r \in O\} \simeq \Pi$ with the following defining relations:

(a) $\widehat{T}_i\widehat{T}_j\widehat{T}_i... = \widehat{T}_j\widehat{T}_i\widehat{T}_j...,\ m_{ij}$ factors on each side,
$\widehat{\pi}_r\widehat{T}_i\widehat{\pi}_r^{-1} = \widehat{T}_j$ if $\widehat{\pi}_r(\alpha_i) = \alpha_j$;

(b) $\widehat{T}_iX_b\widehat{T}_i \ = \ X_bX_{\alpha_i}^{-1},\ \widehat{T}_i^{-1}Y_b\widehat{T}_i^{-1} \ = \ Y_bY_{\alpha_i}^{-1}$
if $(b,\alpha_i^\vee) = 1$ for $0 \le i \le n$;

(c) $\widehat{T}_iX_b \ = \ X_b\widehat{T}_i,\ \widehat{T}_iY_b \ = \ Y_b\widehat{T}_i$
if $(b,\alpha_i^\vee) = 0$ for $0 \le i \le n$;

(d) $\widehat{\pi}_rX_b\widehat{\pi}_r^{-1} = X_{\widehat{\pi}_r(b)},\ \widehat{\pi}_rY_b\widehat{\pi}_r^{-1} = Y_{\widehat{\pi}_r(b)},\ r \in O'$.

Note that no relations between X and Y are imposed. We continue using the notation $X_{[b,j]} = X_b q^j$, $Y_{[b,j]} = Y_b q^{-j}$. The element $q^{1/m}$ is treated as a generator which is central. Later, $q^{1/(2m)}$ will be needed.

The relations (a-d) are obviously invariant with respect to the involution

$$\widehat{\varepsilon}:\ X_b \leftrightarrow Y_b,\ \widehat{T}_i \mapsto \widehat{T}_i^{-1} (0 \le i \le n),\ \widehat{\pi}_r \mapsto \widehat{\pi}_r (r \in O). \tag{4.1}$$

Concerning (d), recall that π_r^{-1} is π_{r^*} for $r^* \in O$. The same holds for $u_r = \pi_r^{-1}\omega_r$ and $\widehat{\pi}_r$.

Theorem 4.1. *The group $\mathfrak{B}$ generated by $X, T, \Pi, q^{1/(2m)}$ subject to the relations (i–v) from Definition 1.1 coincides with the quotient of $\widehat{\mathfrak{B}}$ by the relations:*

$$\widehat{T}_0 = q^{-1}\, X_\vartheta \widehat{T}_{s_\vartheta} Y_\vartheta^{-1},\ \widehat{\pi}_r = q^{(\omega_r,\omega_r)/2}\, Y_r \widehat{T}_{u_r}^{-1} X_{r^*}^{-1}. \tag{4.2}$$

The map is

$$\mathfrak{pr}:\ X_b \mapsto X_b,\ Y_b \mapsto Y_b,\ \widehat{T}_i \mapsto T_i\,(i>0),\ \widehat{T}_0 \mapsto q^{-1}\, X_\vartheta T_{s_\vartheta} Y_\vartheta^{-1},$$
$$\widehat{\pi}_r \mapsto q^{(\omega_r,\omega_r)/2}\, Y_r T_{u_r}^{-1} X_{r^*}^{-1},\ q^{1/(2m)} \mapsto q^{1/(2m)}, \tag{4.3}$$

where the elements $Y_b \in \mathcal{H}$ are given by (1.9). The images of $\widehat{T}_i, \widehat{\pi}_r$ in $\mathfrak{B}$ coincide with $\tau_+(T_i), \tau_+(\pi_r)$ for τ_+ from (1.15). The relations (4.2) are invariant with respect to the involution $\widehat{\varepsilon}$. The latter becomes ε from (1.14) in $\mathfrak{B}$.

Proof. The key fact here is that

$$\varepsilon(\tau_+(T_i)) = (\tau_+(T_i))^{-1}\,(i \ge 0),\ \varepsilon(\tau_+(\pi_r)) = \tau_+(\pi_r), \tag{4.4}$$
$$\text{where } \tau_+(T_i) = T_i \text{ for } i>0,\ \tau_+(T_0) = q^{-1}\, X_\vartheta T_{s_\vartheta} Y_\vartheta^{-1},$$
$$\tau_+(\pi_r) = q^{\frac{(\omega_r,\omega_r)}{2}}\, Y_r T_{u_r}^{-1} X_{r^*}^{-1},\ r \in O.$$

See formula (2.17) for the action of the involution $\eta = \varepsilon\tau_-^{-1}\tau_+\tau_-^{-1}$ from [C9]. We note that (2.17) there directly results in the invariance of the Gaussians with respect to the difference Fourier transform corresponding to the involution ε. The elements $\tau_+(T_i), \tau_+(\pi_r)$ are exactly the images of the elements $\widehat{T}_i, \widehat{\pi}_r$ in $\mathfrak{B}$ under $\mathfrak{pr}$. Relations (4.4) readily follow from the explicit formulas. $\square$

There are important quotients of the group $\widehat{\mathfrak{B}}$ and the algebra $\widehat{\mathcal{H}}$ (see below) obtained by imposing the commutativity of X with Y. They are essential in the theory of the difference Fourier transform.

The $\{X_i\}$ are treated as the coordinates, $\{Y_i\}$ play the role of spectral parameters. Note the immediate projection of these quotients to $\mathfrak{B}$, $\mathcal{H}$ when we make $Y_b = X_b^{-1}$.

Comment. The realization of $\mathfrak{B}$ as a quotient of $\widehat{\mathfrak{B}}$ can be used for a direct proof that ε can be extended to an involution of $\mathfrak{B}$. It can simplify the straightforward proof due to Macdonald and the author from [M], but the difference is not very significant. Our $\widehat{\mathfrak{B}}$ has something in common with the "triple affine Artin group" introduced recently by Ion and Sahi in [IS] for the purpose of interpreting the projective action of $PSL_2(\mathbb{Z})$. Compare our relation (4.2) and formula (23) in [IS] which establishes the connection of their group with the double affine braid group from [C2]. $\square$

Turning to the Hecke algebras, let us define the *universal affine double Hecke algebra* $\widehat{\mathcal{H}}$ as the quotient of the group algebra $\mathbb{Q}_{q,t}\widehat{\mathfrak{B}}$ by the quadratic relations (o) from Definition 1.1 for $\widehat{T}_i$. Here we do not assume that $t_{\mathrm{lng}} = t_{\mathrm{sht}}$. Recall that the elements X_b and Y_b are entirely independent, so the counterpart of the PBW theorem is that an arbitrary element $\widehat{H} \in \widehat{\mathcal{H}}$ can be uniquely represented as

$$\widehat{H} = \sum_{\widehat{w}\in\widehat{W}} Q_{\widehat{w}} T_{\widehat{w}}, \quad \text{where } Q_{\widehat{w}} \text{ are noncommutative polynomials in } X, Y.$$

For applications to the Dunkl operators, this definition will be needed in the following form. We claim that the algebra $\widehat{\mathcal{H}}$ is generated over $\mathbb{Q}_{q,t}$ by the affine Hecke algebra

$$\widehat{\mathcal{H}} \stackrel{\mathbf{def}}{=\!=} \langle \widehat{T}_i, \widehat{\pi}_r \rangle, \ i \geq 0, \ r \in O,$$

the pairwise commutative $X_b\,(b \in P)$, the pairwise commutative $Y_b\,(b \in P)$, satisfying the relations (d) above with the $\widehat{\pi}_r$, and the Lusztig-type relations

$$\widehat{T}_i X_b - X_{s_i(b)} \widehat{T}_i \ = \ (t_i^{1/2} - t_i^{-1/2}) \frac{X_{s_i(b)} - X_b}{X_{\alpha_i} - 1}, \ 0 \leq i \leq n, \tag{4.5}$$

$$\widehat{T}_i Y_b - Y_{s_i(b)} \widehat{T}_i \ = \ (t_i^{1/2} - t_i^{-1/2}) \frac{Y_{s_i(b)} - Y_b}{Y_{\alpha_i}^{-1} - 1}, \ 0 \leq i \leq n. \tag{4.6}$$

Imposing (4.2), we represent $\mathcal{H}$ as a quotient of $\widehat{\mathcal{H}}$. Note that this definition is compatible with the restriction to the lattices B between Q and P taken instead of P.

We set

$$\widehat{Y}_{b_+} \stackrel{\mathbf{def}}{=\!=} \widehat{\pi}_r \widehat{T}_{i_1}\cdots\widehat{T}_{i_l} \quad \text{for} \quad b_+ = \pi_r s_{i_1}\cdots s_{i_l} \text{ as } b_+\in P_+,\ l=l(b_+),$$

more generally, $\widehat{Y}_{b_+-c_+} = \widehat{Y}_{b_+}\widehat{Y}_{c_+}^{-1}$ for $b_+, c_+ \in P_+$.

Universal Dunkl operators. Given a representation $\widehat{V}$ of $\widehat{\mathcal{H}}$, the general universal Dunkl operators are the images of $\widehat{Y}_b\,(b\in P)$ and $\widehat{\pi}_r\,(r\in O)$ in the $\widehat{\boldsymbol{\mathcal{H}\mathcal{H}}}$-module $\mathcal{I}_{\widehat{V}}$ induced from $\widehat{V}$. As a linear space, it is isomorphic to the linear space of *noncommutative* polynomials of X, Y with the (right) coefficients in $\widehat{V}$: $\mathcal{I}_{\widehat{V}} = \cup_{e\in\mathbb{N}}\mathcal{P}_e$ for

$$(4.7)\quad \mathcal{P}_e \stackrel{\mathbf{def}}{=\!=} \Big\{\sum_{\mathbf{b},\mathbf{c}\in\mathbf{P}} X_{b_1}Y_{c_1}\cdots X_{b_e}Y_{c_e}\widehat{v}_{\mathbf{b},\mathbf{c}}\Big\},\ \mathbf{b}\in\mathbf{P}=P^e\ni\mathbf{c},\ \widehat{v}_{\mathbf{b},\mathbf{c}}\in\widehat{V}.$$

Here the sums in (4.7) represent different vectors in $\mathcal{I}_{\widehat{V}}$ for different v-coefficients if we assume that $b_i\neq 0\neq c_j$ for the indices $1<i\le e$, $1\le j<e$ as $\widehat{v}_{\mathbf{b},\mathbf{c}}\neq 0$.

The action of X and Y is by the left multiplication. The subspaces $\mathcal{P}_e$ are $\widehat{\mathcal{H}}$-submodules and also X-submodules for $e>0$.

The action of $\widehat{T}_i\,(i\ge 0)$ and $\widehat{\pi}_r$ in $\mathcal{P}_e$ can be calculated using (4.5),(4.6), and the relations (d) above.

In the first interesting case $e=1$, it is as follows:

$$\begin{aligned}(4.8)\quad \widehat{T}_i(X_bY_c\widehat{v}) \;&=\; X_{s_i(b)}Y_{s_i(c)}\widehat{T}_i(\widehat{v})+\\ &(t_i^{1/2}-t_i^{-1/2})\Big(\frac{X_{s_i(b)}-X_b}{X_{\alpha_i}-1}Y_c + X_{s_i(b)}\frac{Y_{s_i(c)}-Y_c}{Y_{\alpha_i}^{-1}-1}\Big)\,\widehat{v},\ 0\le i\le n,\\ (4.9)\quad \widehat{\pi}_r(X_bY_c\widehat{v}) \;&=\; X_{\pi_r(b)}Y_{\pi_r(c)}\widehat{\pi}_r(\widehat{v}) \quad \text{for}\quad b,c\in P,\ \widehat{v}\in\widehat{V}.\end{aligned}$$

When the initial representation $\widehat{V}$ is the character

$$\widehat{\varpi}_+ : \widehat{T}_i\mapsto t_i^{1/2},\ i\ge 0,\ \widehat{\pi}_r\mapsto 1,$$

we come to a double variant of formulas (1.12) and the corresponding Dunkl operators. Namely:

$$(4.10)\qquad \widehat{T}_i \;=\; t_i^{1/2}s_i^x s_i^y + (t_i^{1/2}-t_i^{-1/2})\Big(\frac{s_i^x-1}{X_{\alpha_i}-1} + s_i^x\frac{s_i^y-1}{Y_{\alpha_i}^{-1}-1}\Big).$$

Here s_i^x, s_i^y act respectively on X and Y, the differences are applied before the division in the divided differences. Similarly, $\widehat{\pi}_r = \pi_r^x\pi_r^y$.

Double polynomials. The above consideration leads to the formulas for the action of the hat-operators in the $\mathcal{H}$-module $\mathbb{Q}_{q,t}[X,Y]$ induced from the character ϖ_+ of the nonaffine Hecke subalgebra $\mathbf{H}$.

These operators are the images of $\{\widehat{T}_i, \widehat{\pi}_r\}$ under the projection $\mathfrak{pr}$. They coincide with $\tau_+(T_i), \tau_+(\pi_r)$. See (4.4). We will use the same hat-notation for them, although now they are the elements of $\mathcal{H}$.

The formulas are:

$$
\begin{aligned}
&\widehat{T}_i(X_bY_c) = X_{s_i(b)}Y_{s_i(c)}\widehat{T}_i(1)+\\
&(t_i^{1/2}-t_i^{-1/2})\Big(\frac{X_{s_i(b)}-X_b}{X_{\alpha_i}-1}Y_c + X_{s_i(b)}\frac{Y_{s_i(c)}-Y_c}{Y_{\alpha_i}^{-1}-1}\Big),\\
&\widehat{T}_i(1) = t_i^{1/2}\ (i>0),\ \widehat{T}_0(1) = q^{-1}\, X_\vartheta T_{s_\vartheta} Y_\vartheta^{-1}(1)\\
&\ = q^{-1}\, X_\vartheta (Y_\vartheta T_{s_\vartheta}^{-1}(1) - (t_0^{1/2}-t_0^{-1/2})),\\
&T_{s_\vartheta}^{-1}(1) = t_{\mathrm{sht}}^{1-(\vartheta,\sum_{\mathrm{sht}}\alpha^\vee)}\, t_{\mathrm{lng}}^{-(\vartheta,\sum_{\mathrm{lng}}\alpha^\vee)},\ \alpha\in R_+.
\end{aligned}
\tag{4.11}
$$

The X-monomials act by the left multiplication, the operators $\widehat{\pi}_r$ via the relations (d) and the formula for $\widehat{\pi}_r(1)$ similar to that from (4.11). Knowing the action of $X_b, \widehat{T}_i$, and $\widehat{\pi}_r$ is sufficient for determining the structure of the $\mathcal{H}$-module.

Recall that the involution ε acts naturally in $\mathbb{Q}_{q,t}[X,Y]$ sending $\widehat{T}_i \mapsto \widehat{T}_i^{-1}$ (all i) and $\widehat{\pi}_r \mapsto \widehat{\pi}_r$.

We note that the hat-operators can be used, for instance, to introduce the "double radial parts". As usual, the simplest ones corresponding to the miniscule symmetric monomial functions, can be calculated explicitely.

To conclude, we note that there is an interesting group of automorphisms of $\widehat{\mathfrak{B}}, \widehat{\mathcal{H}}$ generated by the tau-automorphisms

$$\tau_\pm^x,\quad \tau_\pm^y \stackrel{\mathbf{def}}{=\!=} \widehat{\varepsilon}\tau_\mp^x\widehat{\varepsilon}.$$

They act respectively in $\langle \widehat{T}, \widehat{\pi}, X\rangle$ fixing Y, and in $\langle \widehat{T}, \widehat{\pi}, Y\rangle$ fixing X. This group is an extension of the $PSL_3(\mathbb{Z})$. Hopefully it is "naturally" connected with the action of $PSL_3(\mathbb{Z})$ on solutions of some KZB-type equations found by Felder and Varchenko.

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